Cosmological perturbation theory near de Sitter spacetime

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(Dated: April 21, 2008)

We present a gauge invariant argument that a nonlocal measure of second order metric and matter perturbations dominates that of linear fluctuations in its effect on the gravitational field in spacetimes close to the de Sitter solution.

Introduction — It is well known in the mathematical physics community that linear cosmological perturbation theory about maximally symmetric, spatially closed, spacetimes has peculiar features. Linear perturbations about such special backgrounds must obey certain non-local identities (often discussed under the rubric of linearization stability [5], [6]) which occur at second order in perturbation theory.

In this Letter we study a surprising consequence of these nonlocal identities for perturbations about a slowly-rolling (inflating) spacetime. We find that during 'slow-roll' inflation a certain nonlocal measure of second order metric and matter perturbations generically dominates in its amplitude compared to that of the linear order perturbations, if these identities hold. This provides robust support for the conclusions of one of our previous papers [2], where we found that during slow-roll second order fluctuations grew large for a class of inflationary models. We conclude that is quite plausible that nonlinear, and probably nonperturbative, gravitational effects dominate near de Sitter spacetime (i.e. slow-roll) and therefore linear perturbation theory likely fails in those situations.

Background model — Consider a FRW spacetime in comoving coordinates $(t, \vec{x})$ with scale factor $a(t)$, with signature (-1,1,1,1), and with a perfect fluid with energy density ρ and pressure p. The equations of motion for the scale factor $a(t)$ are, according to the Einstein equations,

$$\frac{\ddot{a}}{a} = -\frac{\kappa}{3}\left[\rho(1+3w) - \Lambda\right], \tag{1}$$

$$H^2 = \frac{\kappa}{3}(\rho + \Lambda) - \frac{K}{a^2}, \tag{2}$$

where $K \equiv \pm 1, 0$ is the constant spatial curvature of the $t = const$ slices, $H \equiv \partial_t ln(a)$ is the Hubble parameter, Λ is a cosmological constant, $w \equiv \frac{p}{\rho}$, and $\kappa \equiv 8\pi G$ in units where $c = 1$.

There are in general six Killing vectors in FRW models, associated with either the $K = 0$ (flat) E(3) rotation group, the $K = -1$ (hyperbolic) SO(3,1) or $K = 1$ (closed) O(4) groups. These three groups are maximal subgroups of the de Sitter group SO(4,1), which has ten parameters corresponding to the six FRW Killing vectors and four boost Killing vectors unique to de Sitter spacetime. The Lie derivative of the FRW metric along four vectors B^a which have the same functional form of de Sitter boost Killing vectors in closed FRW coordinates can be easily calculated, using (1) and (2), to be

$$\pounds_B \bar{g}^{(FRW)}_{ab} = \lambda B^i \delta_{ia}\delta_{0b} \tag{3}$$

where the index i above (and also j, k in what follows) is spatial and 0 refers to the 'time' t, and where

$$\lambda \equiv \frac{1}{2H}\left(-\frac{\kappa\rho}{2}(1+w)\right) \rightarrow 0 \tag{4}$$

as one approaches de Sitter spacetime. Note that the four vectors B^a are merely conformal isometries of the closed FRW spacetime.

Nonlocal constraints — Consider the field equations for scalar matter and metric fluctuations for the above closed FRW solution, near de Sitter spacetime. Assuming that the background matter sector (i.e. ρ, p in equations (1), (2)) is a minimally coupled ('potential dominated') spatially homogeneous scalar field $\bar{\phi}$ with potential $V(\bar{\phi})$, a necessary requirement is that the fluctuations satisfy, order by order in perturbation theory, the initial value constraints on a constant time (spatially compact) hypersurface Σ_t

$$\mathcal{H}_\perp \equiv \mathcal{H}_\perp(h_{ij}, \pi^{ij}, \phi, \pi_\phi) = 0 \tag{5}$$

$$\mathcal{H}^i \equiv \mathcal{H}^i(h_{ij}, \pi^{ij}, \phi, \pi_\phi) = 0 \tag{6}$$

where equations (5) and (6) denote the usual 'Hamiltonian' and 'momentum' constraints respectively for the three metric h_{ij}, its conjugate momentum density π^{ij}, and the scalar field ϕ and its conjugate momentum π_ϕ. These constraints must hold order by

order in perturbation theory for a consistent power series approximation to exist, if one does, for a full solution to Einstein's equations. Also since the constraints hold at each point in space, they must also hold when averaged with arbitrary functions over space.

Consider a projection (average) of these constraints along an arbitrary 4-vector field X. Denoting this by $P(X)$, we write

$$P(X) \;\equiv\; \int_{\Sigma_t} \left(X_\perp \mathcal{H}_\perp + X_\parallel^i \mathcal{H}_i \right) d^3x = 0, \quad (7)$$

where $X^a \;\equiv\; X_\perp \bar{n}^a + X_\parallel^a \bar{h}^j{}_a$is a four dimensional vector field where n^a is the normal to Σ_t. We wish to approximate $P(X)$ order by order in perturbation theory.

Given a quantity q we will designate the first order variation by δq and the second order by $\delta^2 q$. Furthermore we will designate the background quantities by an overbar $\bar{q}$. If we consider variations in h_{ij} and π^{ij} (along with ϕ, π_ϕ), we can calculate the corresponding classical variation in $P(X)$. We demand that the background quantities obey the full Einstein equations. Using Hamilton's equations for the background to *define* the time derivatives $\dot{\pi}^{ij}$, $\dot{h}_{ij}, \dot{\phi}, \dot{\pi}_\phi$ on a spatially compact Σ_t, one can show that [11]

$$\begin{aligned} \delta P(X) \;=\; & \int_{\Sigma_t} \left[\left(\pounds_X \bar{h}_{ij} \right) \delta\pi^{ij} - \left(L_X \bar{\pi}^{ij} \right) \delta h_{ij} \right. \quad (8) \\ & \left. + \left(\pounds_X \bar{\phi} \right) \delta\pi_\phi - \left(\pounds_X \bar{\pi}_\phi \right) \delta\phi \right] d^3x, \end{aligned}$$

where $\pounds_X \bar{h}_{ij}$ is the spatial restriction of $\pounds_X g_{ab}$ to the (spatially compact) hypersurface Σ_t:

$$\pounds_X \bar{h}_{ij} \;=\; \pounds_{X_\parallel} \bar{h}_{ij} + X^0 \dot{\bar{h}}_{ij} + 2\bar{N}_{(i} X^0_{|j)}, \quad (9)$$

where $\bar{N}_i \;\equiv\; \bar{g}_{0i}$ is the 'shift vector' (here and in what follows all barred quantities will be background quantities). The calculations are simplified if we take the **background** values of $\bar{N}_i = 0$ and $\bar{N} \equiv -\bar{g}_{00} = 1$.

To give an idea how this is derived, consider the variation with respect to $\delta\phi$. One of the terms in the above is

$$\begin{aligned} & \int_{\Sigma_t} X^0 \sqrt{|\bar{h}|} \left(h^{ij} \partial_i \bar{\phi} \partial_i \delta\phi + V'(\bar{\phi}) \delta\phi \right) d^3x \\ = & \int_{\Sigma_t} X^0 \dot{\bar{\pi}} \delta\phi d^3x \quad (10) \end{aligned}$$

because $\partial_i \bar{\phi} = 0$ in the background. Similarly,

$$\int_{\Sigma_t} X^0 \frac{1}{\sqrt{|\bar{h}|}} \bar{\pi}_\phi \delta\pi_\phi d^3x = \int_{\Sigma_t} X^0 \dot{\bar{\phi}} \delta\pi d^3x \quad (11)$$

Finally, using the metric equations of motion of the background spacetime, we can show that

$$\begin{aligned} L_X \bar{\pi}^{ij} \;=\; & X^0 \dot{\bar{\pi}}^{ij} + \pounds_{X_\parallel} \bar{\pi}^{ij} \\ & + \sqrt{|\bar{h}|} (\bar{D}^i \bar{D}^j - \bar{h}^{ij} \bar{D}_k \bar{D}^k) X^0, \quad (12) \end{aligned}$$

where $\bar{D}_i$ is the induced covariant derivative on Σ_t. Putting equations (9) through (12) into equation (8) we finally obtain the general expression for the linearized projection of the initial value constraints, $\delta P(X)$.

If we take the de Sitter limit, i.e. $V_{,\bar{\phi}}, \dot{\bar{\phi}} \to 0$, $\dot{H} \to 0$, then $\pounds_X \bar{h}_{ij} \to 0$ and $L_X \bar{\pi}^{ij} \to 0$ (and similarly for the matter fluctuations) yields $\delta P(X) \to 0$. Thus, as is well known, the linearized projection of the constraint equations is identically zero along a Killing direction of the background spacetime provided the matter fluctuations obey the equations of motion ([11], [12], [7], [6]).

The second order equations now have the form

$$\begin{aligned} \delta^2 P(X) \;=\; & \int_{\Sigma_t} \left[\left(\pounds_X \bar{h}_{ij} \right) \delta^2\pi^{ij} - \left(L_X \bar{\pi}^{ij} \right) \delta^2 h_{ij} \right. \\ & \left. + \left(\pounds_X \bar{\phi} \right) \delta^2\pi_\phi - \left(\pounds_X \bar{\pi}_\phi \right) \delta^2\phi \right] d^3x \\ & + O(\delta q \delta q) \quad (13) \end{aligned}$$

where the last term represents all of the terms quadratic in the first order perturbations. This implies that in looking at the *second* order projection along the Killing vector(s), the terms linear in the second order perturbations is zero, and the nontrivial quadratic term must also be zero. This represents an additional constraint on the first order perturbations which must be set to zero if the second order equations are to be satisfied.

However as we discussed above it is clear that *near* a de Sitter spacetime one does not have exact *boost* symmetries. If one projects the linearized constraint denisities $\mathcal{H}^\alpha$ along vectors B^a which have a de Sitter boost *functional form* in closed FRW coordinates, as described above, then using equations (3) it follows from equations (8)-(12) that

$$\delta P(B) \propto \frac{\lambda}{H} \neq 0, \quad (14)$$

so that in the de Sitter limit, ($\lambda \to 0$) the Killing identity is recovered.

At second order in perturbation theory, we thus expect that $\delta^2 P(B)$ has *two* terms: one additional second order term multiplied by $\frac{\lambda}{H}$ and the quadratic piece. We compute $\delta^2 P(B)$ in the present Hamilto-

nian formalism and find

$$\begin{aligned}\delta^2 P(B)_{\Sigma_t} &= \int_{\Sigma_t} B^a \delta^2 \mathcal{H}_a d^3x \qquad (15)\\ &= \int_{\Sigma_t} \left[\left(\pounds_B \bar{h}_{ij}\right)\delta^2\pi^{ij} - \left(L_B \bar{\pi}^{ij}\right)\delta^2 h_{ij}\right.\\ &\quad \left. + \left(\pounds_B \bar{\phi}\right)\delta^2\pi_\phi - \left(\pounds_B \bar{\pi}_\phi\right)\delta^2\phi\right] d^3x\\ &\quad + \int_{\Sigma_t} \left[\left(\pounds_B \delta h_{ij}\right)\delta\pi^{ij} - \left(L_B \delta\pi^{ij}\right)\delta h_{ij}\right.\\ &\quad \left. + \delta\pi_\phi(\pounds_B\delta\phi) - \delta\phi(\pounds_B\delta\pi_\phi)\right] d^3x,\end{aligned}$$

In the special case that B^a *is* a Killing vector, i.e. the background is closed vacuum de Sitter spacetime, it is clear that demanding the right hand side of (14) vanish implies a nontrivial and spatially nonlocal constraint on the linear initial values $(\delta h_{ij}, \delta\pi^{ij}; \delta\phi, \delta\pi_\phi)$. In this case the nonlocal constraint, an integral over a density, is gauge invariant and preserved from slice to slice [7].

Slow-roll limit — It is apparent from equation (14) that there is an overall prefactor of $\frac{\lambda}{H}$ multiplying the second order terms compared to the final product term involving the linear fluctuations. Comparing the two groups of terms, second order ($\equiv \delta^2 P_S(B)$) and quadratic in first order ($\equiv \delta^2 P_Q(B)$), we rework equation (15) by explcitly writing out the linear factor of $\frac{\lambda}{H}$ in $\delta^2 P_S$:

$$\begin{aligned}\delta^2 P(B) &= \left(\frac{\lambda}{H}\right)\delta^2 P_S(B)[\delta^2 q_i] \qquad (16)\\ &\quad + \delta^2 P_Q(B)[(\delta q_i)^2, \delta q_i \delta q_j],\end{aligned}$$

where the entire set of second and linear order canonical variables is written as $\delta^2 q_i$ for the second order fluctuations, ($(\delta q_i)^2, \delta q_i \delta q_j$) denotes the quadratic combinations of the first order fluctuations, and $\delta^2 P_S(B) \equiv \frac{\lambda}{H}\delta^2 \tilde{P}_S(B)$. Thus whenever the slow-roll approximation for the background holds, i.e. $\frac{\lambda}{H} \ll 1$, we may approximately solve equation (16) for $\delta^2 P_S(B)[\delta^2 q_i]$ to find

$$\delta^2 P_S(B)[\delta^2 q_i] \approx -\frac{H}{\lambda}\delta^2 P_Q(B)[(\delta q_i)^2, \delta q_i \delta q_j]. \qquad (17)$$

Thus, this combination of second order terms equals a large number times some combination of the first order term. Assuming that the linear fluctuations are not too small, this implies that at least this combination of the second order fluctuation is larger than the first order perturbations. This is the main result of this paper: that *a nonlocal combination of second order metric and matter fluctuations will generically dominate in its effect on the projection of the gravitational constraints along B^a compared to the linear terms* . Note that if $\delta q_i \ll \lambda/H$ the linearized fluctuations will not have the correct amplitude for seeding CMB fluctuations.

Gauge invariance — Equation (17), the main result of this paper, was derived without assuming a specific gauge choice. We now show that one cannot choose a second and/or linear order gauge such as to eliminate the factor of $\frac{H}{\lambda}$ in equation (17).

Although $\delta P(B) \neq 0$ for any $\lambda \neq 0$, it is easy to show that the *background* projection $\bar{P}(B)$ actually vanishes identically for *any* value of λ, i.e.

$$\bar{P}(B) = 0 \qquad (18)$$

for the background (closed FRW) constraints holding. Thus $\delta^2 P(B)$ cannot depend on any purely second order infinitesimal coordinate transformation, just like any linear perturbation of a background constant is automatically gauge invariant to linear order.

The most general remaining gauge transformation of equation (17) will induce an equation that can be written as

$$F_S(2\pounds_\zeta \delta q_i, \pounds_\zeta{}^2 \bar{q}_i) \approx -\frac{H}{\lambda}F_Q(2\pounds_\zeta \delta q_i, \pounds_\zeta{}^2 \bar{q}_i)(19)$$

where ζ^a is an linearized (infinitesimal) coordinate transformation (so, e.g. $\delta^2\pi^{ij} \to \delta^2\pi^{ij} + \pounds_\zeta^2\bar{\pi}^{ij} + 2\pounds_\zeta\delta\pi^{ij}$) and F_S, F_Q are the gauge terms coming from $\delta^2 P_S$ and $\delta^2 P_Q$ respectively. If one chooses

$$\zeta^a \equiv \left(\frac{\lambda}{H}\right)^n \tilde{\zeta}^a, n \in \mathcal{Z}^+,$$

such that n is the value required to eliminate the factor of $\frac{H}{\lambda}$ then one can rewrite (19) (by decomposing F_Q, F_P into parts linear and quadratic in ζ^a) as

$$\begin{aligned}&\left(\frac{\lambda}{H}\right)^n \left[{}^{(1)}f_S(2\pounds_{\tilde{\zeta}}\delta q_i) + \left(\frac{\lambda}{H}\right)^n {}^{(2)}f_S(\pounds_{\tilde{\zeta}}{}^2\bar{q}_i)\right] \qquad (20)\\ &\approx -\frac{H}{\lambda}\left(\frac{\lambda}{H}\right)^n \left[{}^{(1)}f_Q(2\pounds_{\tilde{\zeta}}\delta q_i) + \left(\frac{\lambda}{H}\right)^n {}^{(2)}f_Q(\pounds_{\tilde{\zeta}}{}^2\bar{q}_i)\right]\end{aligned}$$

which clearly reduces to

$${}^{(1)}f_S(2\pounds_{\tilde{\zeta}}\delta q_i) \approx -\frac{H}{\lambda}{}^{(1)}f_Q(2\pounds_{\tilde{\zeta}}\delta q_i) \qquad (21)$$

given that $\left(\frac{\lambda}{H}\right)^n \sim 0$, which is precisely of the form of equation (19).

In summary, the form of equation (19) must persist given any first and second order gauge fixing in the perturbation theory, including in particular the trivial choice $\zeta^a = 0$. Another way of saying this is that the gauge dependence on both sides of equation (16) acts in such a way as to always preserve the

form of equation (17). This is to be distinguished from the $\lambda = 0$ case, where the constraints (17) are *exactly* gauge invariant to second order.

Quantum anomalies — The quadratic terms in equations (17) formally need to be regularized if we regard them as products of interacting quantum fields (see e.g., [9], [10], [8]). Renormalization ambiguities could imply important quantum anomalies with respect to the imposition of second order conditions such as (17), in addition to any other reasonable conditions such as the conservation of stress-energy.

To begin with, one can show that there *will not* in general be anomalies associated with the simultaneous imposition of stress energy conservation and the equations of motion provided the background spacetime is slowly rolling. This is so because we can, in this very special case, specify the renormalization ambiguities (i.e. the nonuniqueness of a nonlinear monomomial (and its derivatives) in the fields) to absorb the considerably simplified slow-roll curvature counterterms. Specifically, for the case of the scalar field $\delta\phi$ it is known that the monomials $\Psi \equiv (\delta\phi)^2$, $\Psi_{ab} \equiv \bar{\nabla}_a \delta\phi \bar{\nabla}_b \delta\phi$ are unique up to the transformations [8]

$$\Psi \ \rightarrow \ \Psi + C \tag{22}$$

$$\Psi_{ab} \ \rightarrow \ \Psi_{ab} + C_{ab} \tag{23}$$

where C, C_{ab} are quantities constructed from the metric g_{ab}, curvature, and derivatives of the curvature of the appropriate scaling dimension. For slow roll backgrounds, C_{ab} and C have a simple functional form.

Using this simplification one may show, just as we did for the case of pure de Sitter spacetime in [1], that there are no *additional* anomalies associated with the imposition of the purely matter part of (17). This is so because all the anomaly terms are proportional to integrals over Σ_t of $B_a n^a$ (which even for all $\lambda \geq 0$ is spatially odd), which are identically zero. It turns out that *if* the remaining quadratic gravitational terms in (17) can be cast as quadratic scalar field terms (with some technical qualifications related to eliminating the homogeneous and dipole modes), where the scalar fields represent polarizations of the transverse traceless excitations δh_{ij}, $\delta\pi^{ij}$ [4] and the lone scalar mode at linear order, then same logic goes through as for the scalar field case. One would then conclude, remarkably, that there are no additional quantum anomalies associated with the imposition of relations (17). However, it should be strongly emphasized that in the absence of an explicit expression for the tensorial anomalies this is at best a plausible assumption (see [3]).

Conclusions — In a previous publication [2] we have observed second order effects becoming large in some slow-roll models, however the present argument is demonstrably gauge invariant to second order and only essentially relies on the assumptions that the constraints are satisfied order by order in perturbation theory and that de Sitter spacetime has boost Killing vectors. Although our constraint analysis cannot answer the dynamical question of *when* (i.e. after how many e-foldings) these higher order effects can be expected to make a difference in typical slow-roll inflation or other models, our claim is that the worrisome higher order effects *do* unambiguously and rather generically enter with very minimal assumptions - they are really there. We hope that by sidestepping the usual costly debate over whether or not higher order perturbative effects are just gauge effects or other artifacts of poorly controlled approximations, the present argument will serve as further motivation to probe higher order effects in cosmological perturbation theory near de Sitter spacetime.

Acknowledgements — W.G.U. would like to thank the Canadian Institute for Advanced Research and NSERC for support. B.L. acknowledges YITP Workshop YITP-W-07-10 (and YITP computer facilities) for hospitality and support, NSERC for support, and R. Brandenberger, E. Mottola, M. Sasaki, T. Tanaka, and R. Woodard for many useful conversations. We also thank two anonymous referees for valuable suggestions and criticism.

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[1] B. Losic and W. G. Unruh, Phys. Rev. D **74**, 023511 (2006) [arXiv:gr-qc/0604122].
[2] B. Losic and W. G. Unruh, Phys. Rev. D **72**, 123510 (2005) [arXiv:gr-qc/0510078].
[3] B. Losic, J. Phys. A **40**, 6951 (2007).
[4] L. H. Ford and L. Parker, Phys. Rev. D **16**, 1601 (1977).
[5] A. E. Fischer and J. E. Marsden, *Gen. Rel. Grav.*, **7**:915–920, 1976.
[6] D. Brill and S. Deser, Commun. Math. Phys, **32**:291-304 (1973)
[7] V. Moncrief, Phys. Rev. D **18**, 983 (1978).
[8] S. Hollands and R. M. Wald, Rev. Math. Phys. **17**, 227 (2005) [arXiv:gr-qc/0404074].
[9] Stefan Hollands and Robert M. Wald. *Commun. Math. Phys.*, 237:123–160, 2003.
[10] Stefan Hollands and Robert M. Wald. *Commun. Math. Phys.*, 231:309–345, 2002.
[11] A. Higuchi, Class. Quant. Grav. **8**, 1961 (1991).
[12] V. Moncrief, *J. Math. Phys.*, **17**:1893–1902 (1976).